
Wake Fields and Beam Dynamics

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Overview

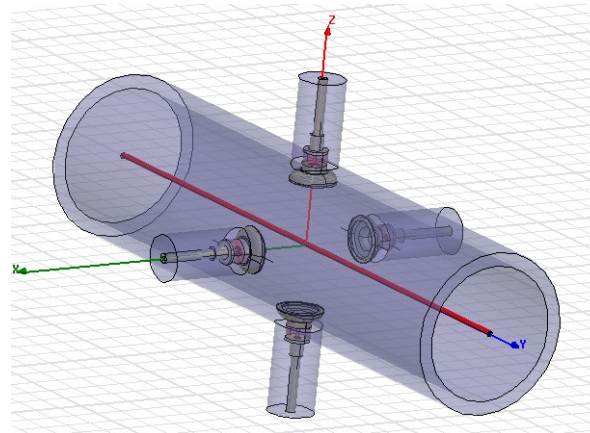
- A study of how particles affect other particles in accelerators:
 - The importance of wake fields in storage rings.
 - The effect on the stable movement of particles.
 - Computing wake fields in cylindrical beam pipes.
 - Equations of motion of particles in wake fields.
 - Simulation of particle movements.
 - Future work and challenges.

Wake Fields in Storage Rings

- Storage rings are circular accelerators in which charged particles circulate.
- These can be used to generate synchrotron radiation, or produce high quality beams for high energy physics experiments.
- These particles generate electric fields (wake fields) that perturb the stable particles behind.
- In new generations of accelerators (CLIC, ILC, ...), it is getting increasingly difficult to suppress the growing instabilities.
- Accurate predictions are needed in order to control them.

The Nature of Wake Fields

- In a real accelerator:
 - Wake fields are determined by the geometry and materials of the environment around the beam.

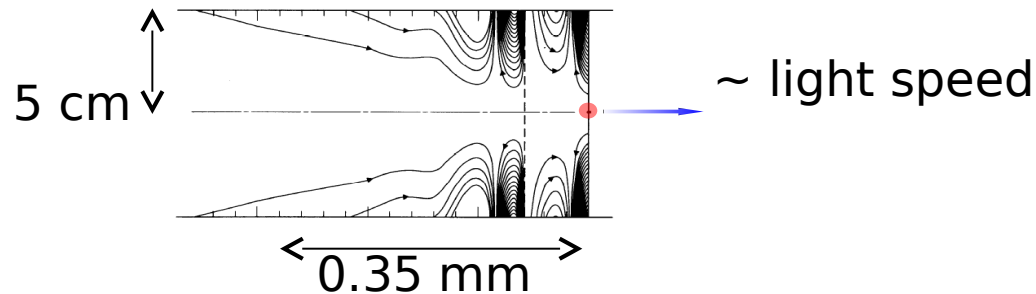


e.g. Beam Position Monitors
Generate complex wake fields

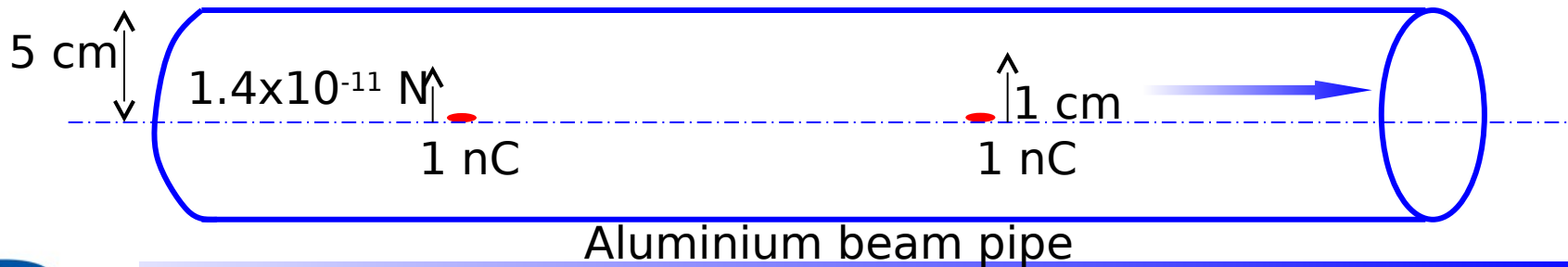
- Short-range effects act within a single bunch, and long-range effects act between bunches.
- The current work focuses on the uniform beam pipe and long range effects.

Wake Field from the Beam Pipe

Wake field is the electric field generated by a charged particle moving close to light speed. There is no field in front because signal cannot travel faster than light.

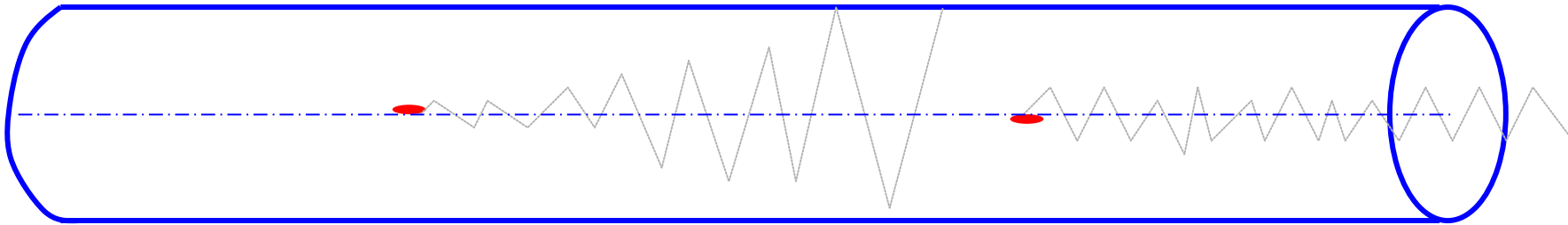


Charged particles are often compressed into bunches. The example below is for 1 GeV electrons. The wake force looks small but, given time, is enough to destabilise the bunch.



The Problem of Wake Fields

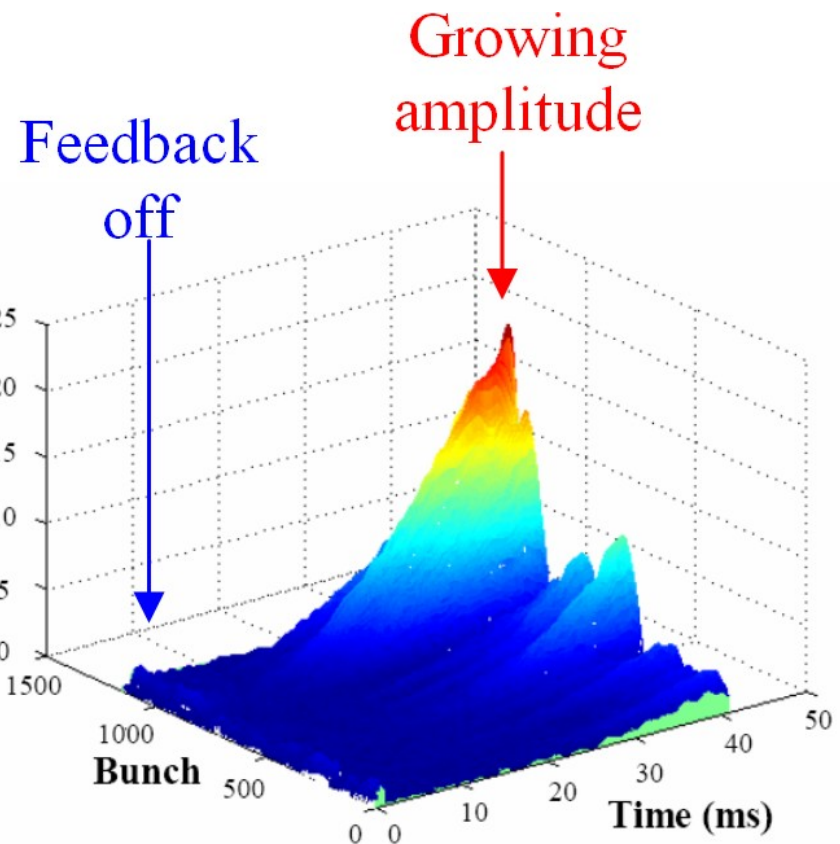
Bunches coupled by wake field can oscillate and grow in amplitude. If this oscillation is not suppressed, the bunches will hit the wall after some time.



- In the ILC damping ring (6 km, 5 GeV, 400 mA, 3 cm radius, 61.4 tune), oscillation grows by 0.7% per turn.
- For an initial oscillation of $1\text{ }\mu\text{m}$, the bunches will hit the wall in 0.03 seconds

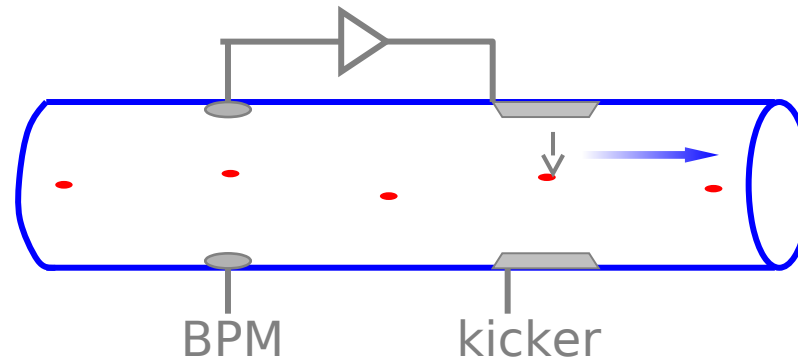
Growing oscillations observed

- In the KEKB electron ring in Japan, the oscillations are normally controlled by a feedback system.
- When the feedback is switched off, the oscillation amplitudes are observed to grow exponentially.



Method to control the oscillations

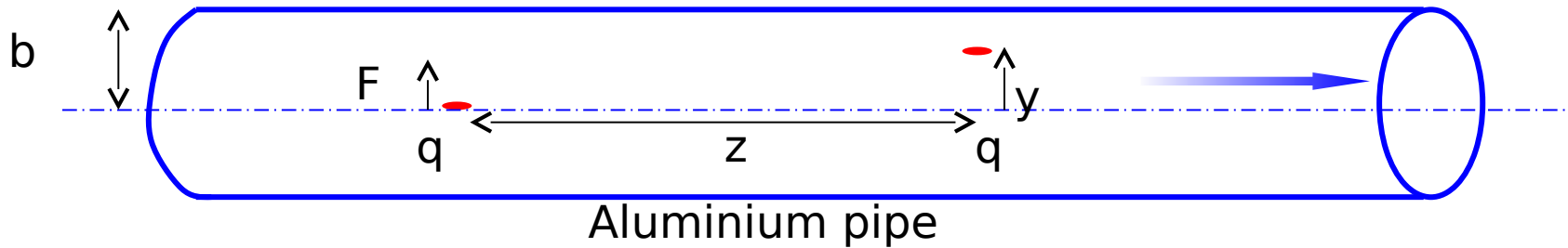
- Schematic of a feedback system



- Beam position monitor (BPM) detects position of bunch, and an electronic system processes the signal
 - Kicker provides a deflection to bring the bunch back towards the desired trajectory
 - The kick takes place after one turn, since the signal cannot travel faster than light.
- New accelerators require very fast response. This pushes the system to the limit of technology.

Transverse Wake Force

- Assuming that coupling between longitudinal and transverse motion is weak, only the transverse wake force is required here:

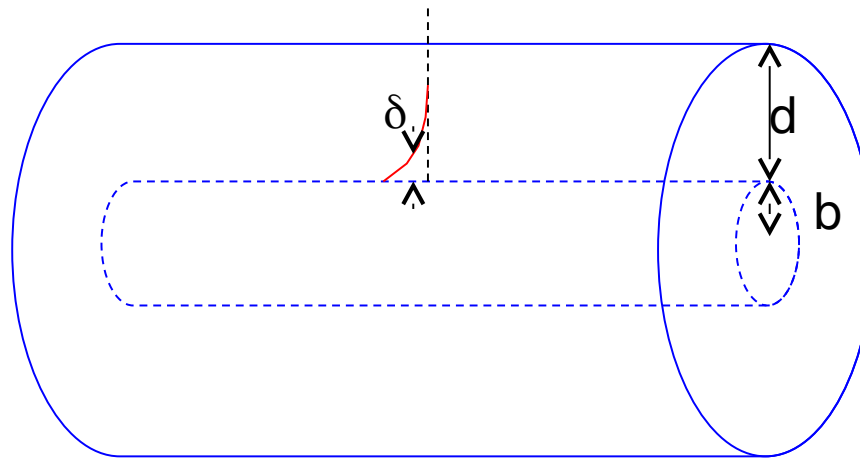


$$F = -q^2 W_1(z) y / L$$

- q is the charge of each bunch
- L is the circumference of the ring
- $W_1(z)$, the wake function, depends on the size and material of the beam pipe

Assumptions to simplify problem

- Analytic calculation is possible if the following assumptions are made (Chao 1993):
 - the wall of the beam pipe is infinitely thick
 - the skin depth is very short compared with the radius.



$$d \gg \delta$$
$$b \gg \delta$$

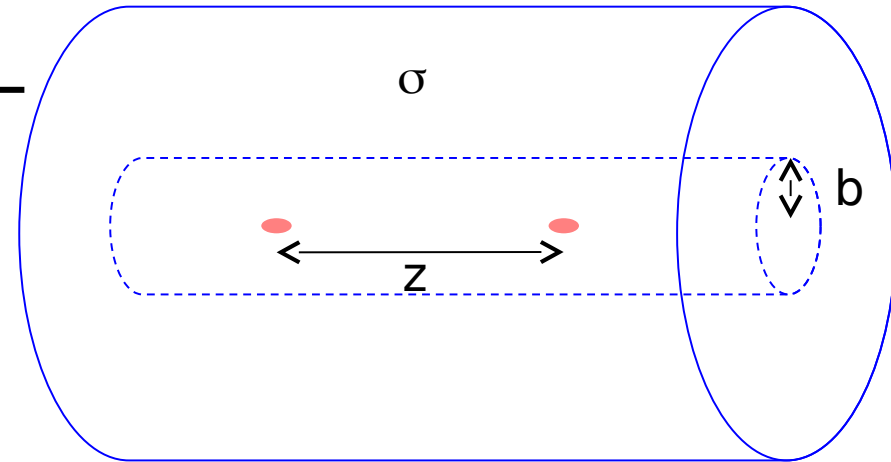
- Both assumptions are true in the high frequency limit, but are often far from reality. Nevertheless, the formulae produced have been in use for decades.

An approximate analytic formula

- Assuming the high frequency limit, the transverse wake function is:

$$W_1(z) = -2/(\pi b^3) \sqrt{c/\sigma z} L$$

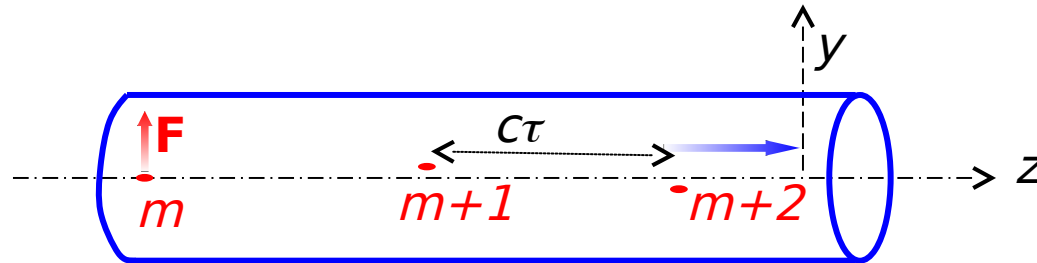
- b = beam pipe radius
- σ = wall conductivity
- z = bunch spacing



- The z dependence has the simple form of $1/\sqrt{z}$

Equations of motion

- Assume that the focusing strength is uniform. Each bunch experiences two types of forces:
 - the focusing force from the magnets, $-M\omega_\beta^2 y$
 - ω_β is the betatron frequency,
 - M is the mass of the bunch
 - sum of the wake forces from bunches in front.



- The equation of motion for each bunch is then:

$$M d^2 y_m / dt^2 = -M \omega_\beta^2 y_m - q^2 / L [W_1(-c\tau) y_{m+1} + W_1(-2c\tau) y_{m+2} + \dots]$$

where y_m the displacement of bunch m

Comparison with experiments

- When the approximate wake function is used with the equations of motion, the growth rates of the oscillation can be calculated analytically.

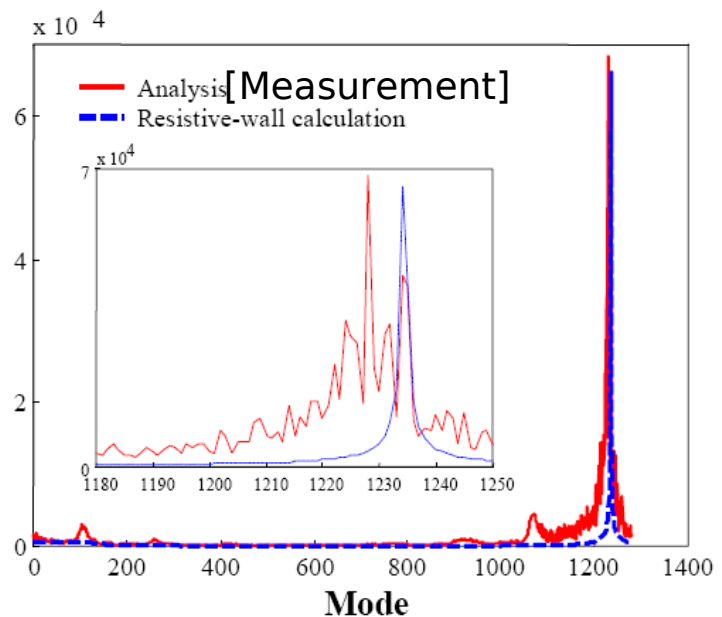
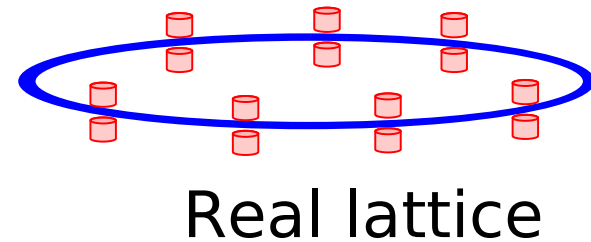
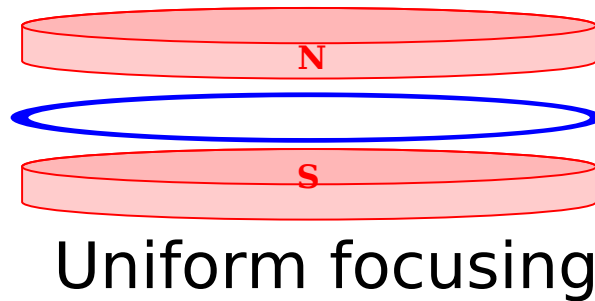


Fig. 1 The horizontal mode distribution of HER at 700 mA

- Measurements in KEKB shows that trends agree. However, significant errors remain.

To improve the standard model

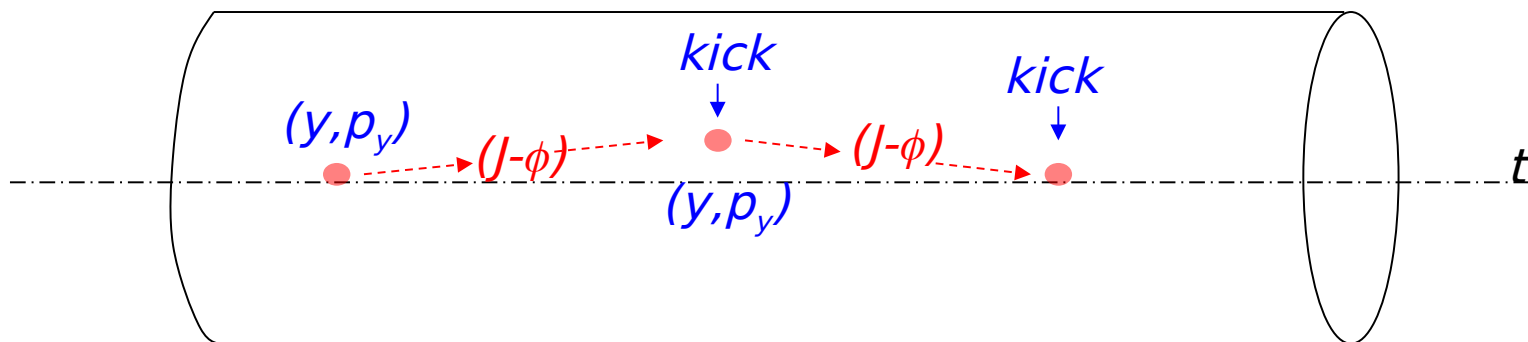
- The assumptions used in the model are:
 - uniform focusing strength,
 - high frequency limit (thick beam pipe, small skin depth).



- To improve the model, we start by using a real lattice.

Time domain simulation

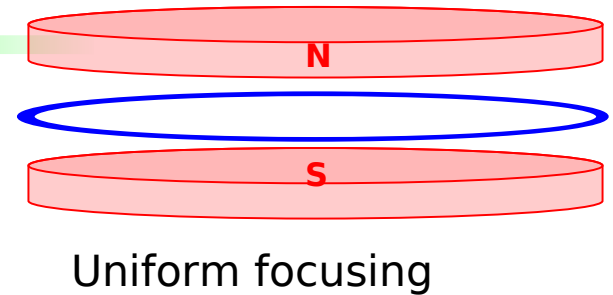
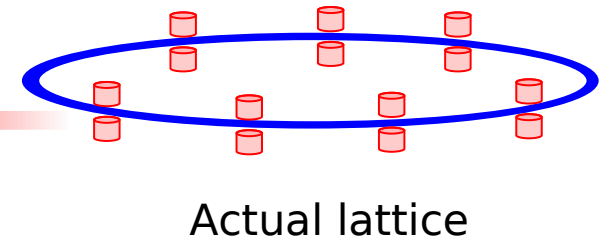
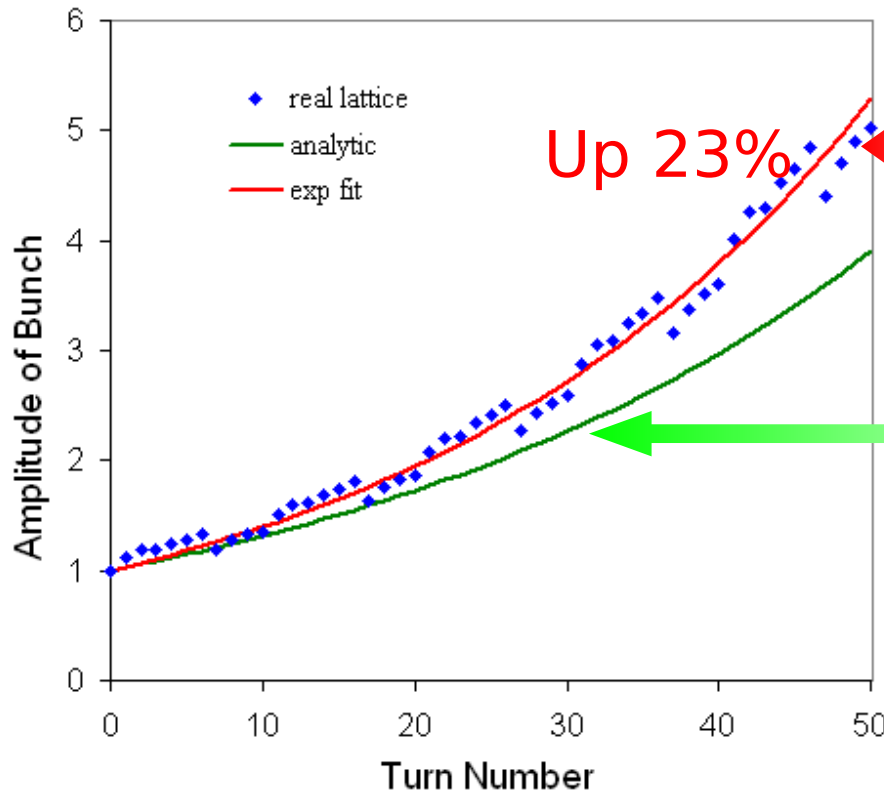
- With a real lattice, analytic solution is no longer possible. To carry out the numerical integration:
 - First, ignore wake force. Obtain the coordinates of each bunch after one time step using the lattice functions (and action-angle variables).



- Then, add the impulse (kick) of the wake force over this time step to the momentum of the bunch.
- To benchmark: Compare with the analytic solution when the focusing strength is uniform. Agreement shows that the method is accurate.

Simulation Results

Growth rate in ILC damping ring



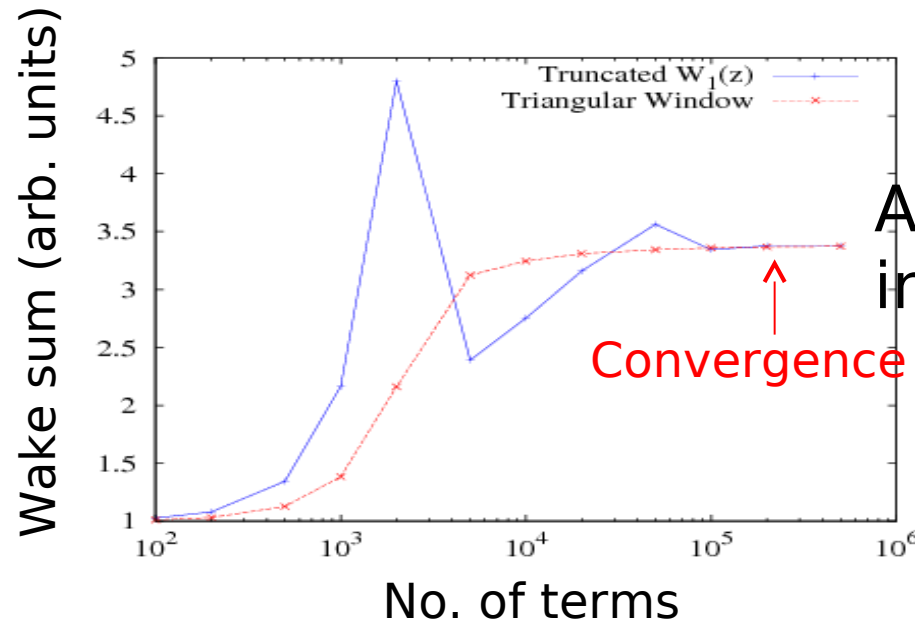
is provides a more accurate specification for feedback system
or further improvement: an accurate wake function is needed

Hock and Wolski, Phys. Rev. ST AB 10, 084401 (2007)

Convergence Issues

$$M d^2 y_m / dt^2 = -M \omega_\beta^2 y_m - q^2 / L [W_1(-c\tau) y_{m+1} + W_1(-2c\tau) y_{m+2} + \dots]$$

- The wake field from each bunch can go round the ring many times. So the sum of wake forces has taken to infinity, and convergence is a problem.



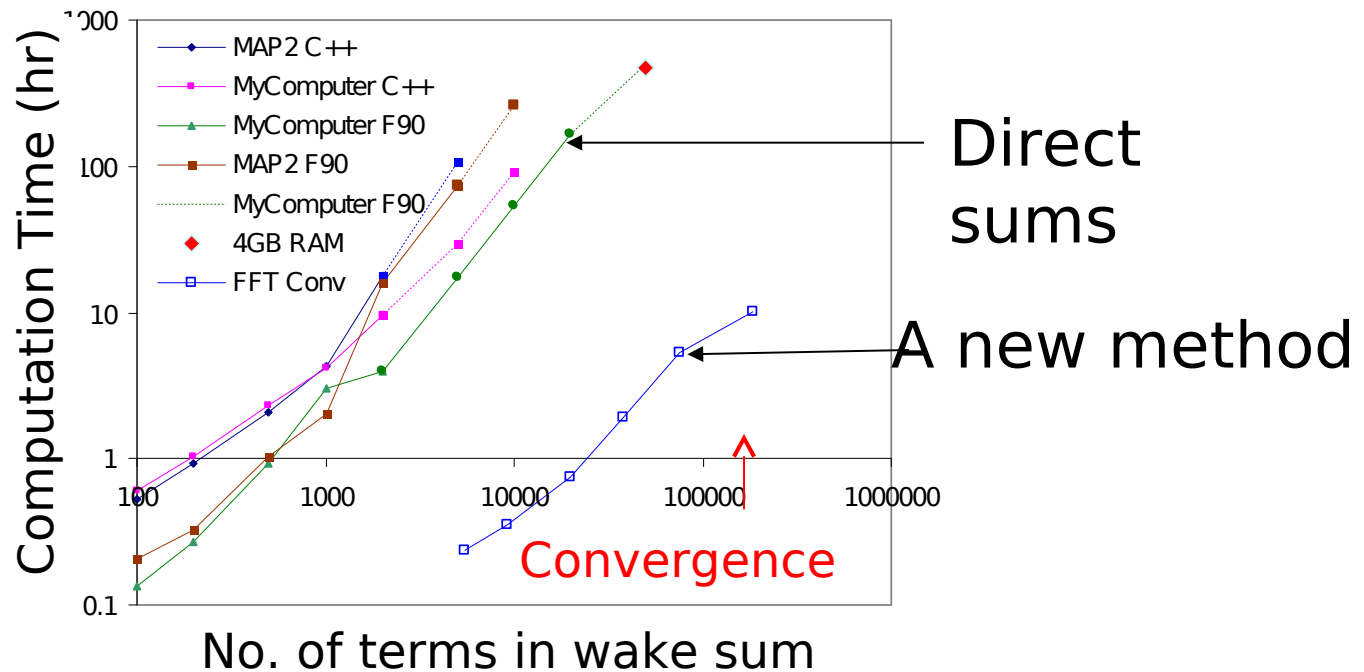
Adding wake forces
in ILC damping ring

- Multiplying the terms by a function (window function) to smooth out the truncation of the sum improves the convergence rate, but a full convergence may still take months.

Wake Sum Computation Times

- Many ways to optimise the computation have been tried: different compilers, different computers, different memory size. None worked.
- A new method is needed.

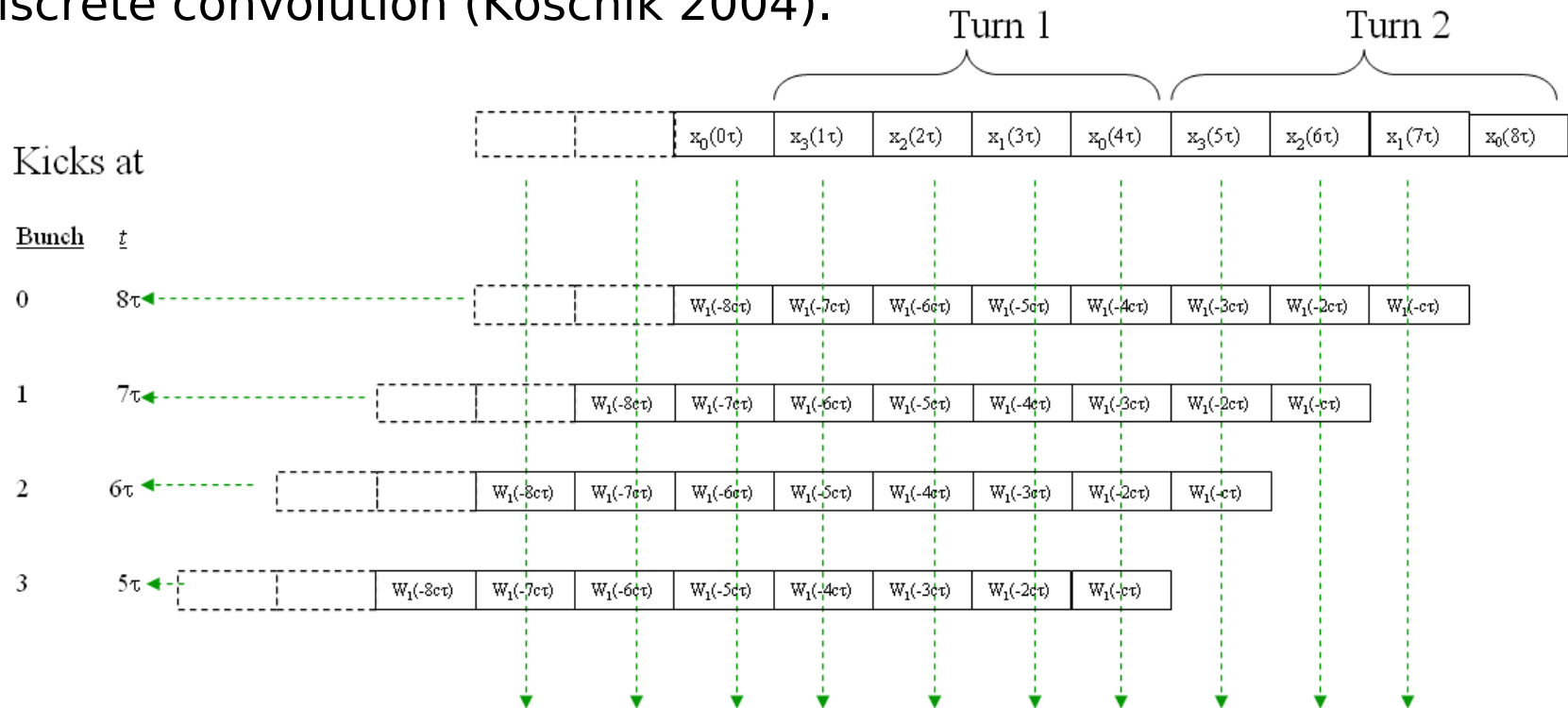
45 Turns on the ILC damping ring



The FFT Convolution Method

$$M d^2 y_m / dt^2 = -M \omega_\beta^2 y_m - q^2 / L [W_1(-c\tau) y_{m+1} + W_1(-2c\tau) y_{m+2} + \dots]$$

- Within each turn, the sum can be arranged into the format of a discrete convolution (Koschik 2004).

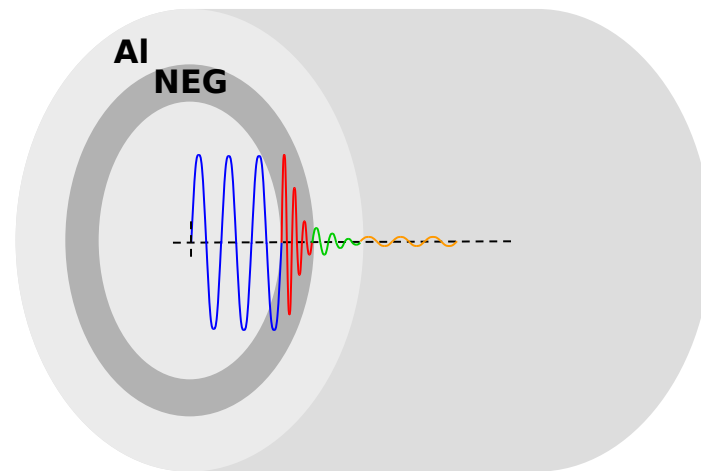


- Applying the convolution theorem, the FFT can then be used to speed up the computation.

In the case of the ILC damping ring, the computation speed is increased by 100 times. Computation can now be done within a day.

A more accurate wake function

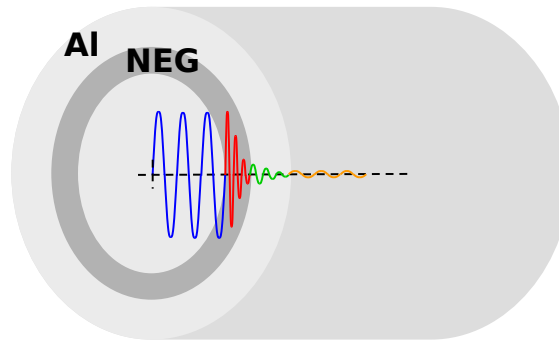
- Next, the assumptions for the wake function – very thick wall and small skin depth – are removed.
- For realistic wall and bunches, a full wave calculation of Maxwell's equations is needed.



- Extra coatings to improve the vacuum may also be present. This could modify the wake function further.

Multilayer beam pipe wall

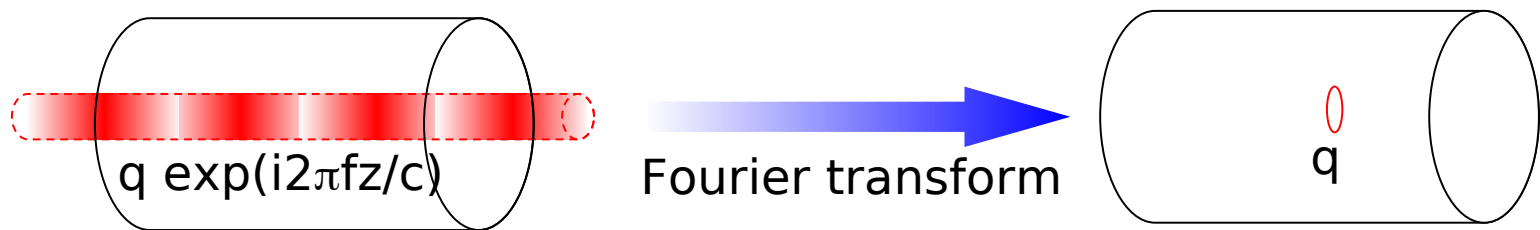
- A NEG coating may be added to the inner wall to improve the vacuum. NEG is an alloy of titanium. (The beam pipe is often made from Aluminium.)



- The electrical resistance of NEG is an order of magnitude higher than Aluminium. This would have an impact on the wake function.
- Grooves and antechambers may also be added to deal with space charge problems. We consider the NEG coating first – its cylindrical symmetry fits into the existing calculation readily.

To find the wake function

- First, Maxwell's equations are solved analytically in the frequency domain, for cylindrical symmetry.
- The beam pipe is assumed to be straight. For results to be valid, radius of the pipe should be small compared with radius of ring.
- The result is the electric field due to a continuous charge distribution along the pipe, with a particular frequency modulation along the axis.



- This is called the impedance. Wake function then is obtained by Fourier transform.

A Full Wave Calculation

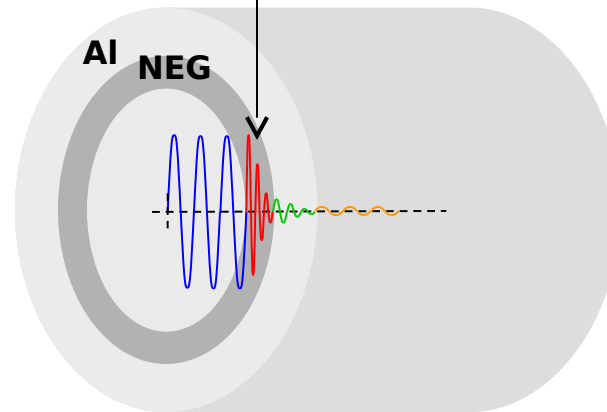
- Analytical solutions for the electric and magnetic fields can be obtained for each layer of the beam pipe.
- These are the two modified Bessel's functions. The field is given by a linear combination of them.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

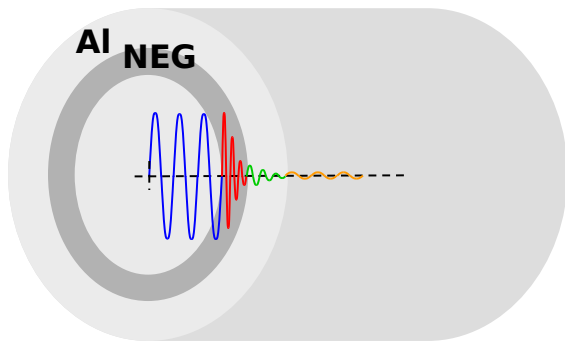
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$



Getting the wake function

- The electric and magnetic field components are then matched at each interface. This gives a linear system of equations.
- The coefficients a_{ij} are formed by the modified Bessel's functions.



$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1N}x_N = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2N}x_N = b_2$$

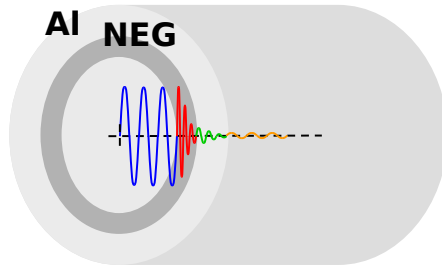
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3N}x_N = b_3$$

...

$$a_{M1}x_1 + a_{M2}x_2 + a_{M3}x_3 + \cdots + a_{MN}x_N = b_M$$

- Solving these equations give the electric field at the axis, which then gives the impedance.

Computational Issues



$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1N}x_N = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2N}x_N = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3N}x_N = b_3$$

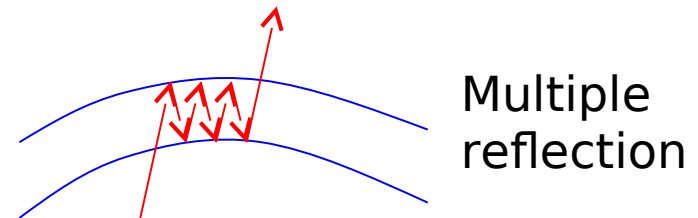
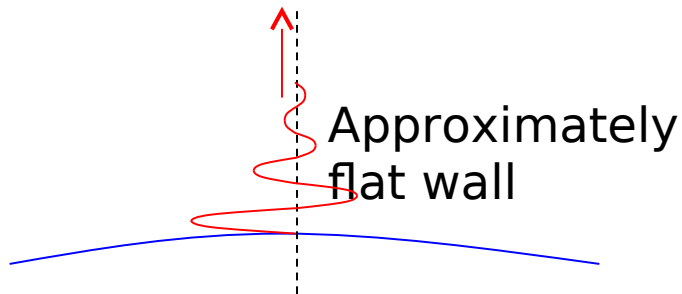
...

$$a_{M1}x_1 + a_{M2}x_2 + a_{M3}x_3 + \cdots + a_{MN}x_N = b_M$$

- To obtain the impedance, the linear system is solved by Gaussian elimination.
- Because of rapid damping of the waves in the wall, some coefficients are extremely small. Numerical precision must be increased using software.
- Even then, computation fails at high frequency as skin depth becomes very small. This is a problem if the wake function is required at a very small distances (e.g. within the same bunch).

Lifting the assumptions

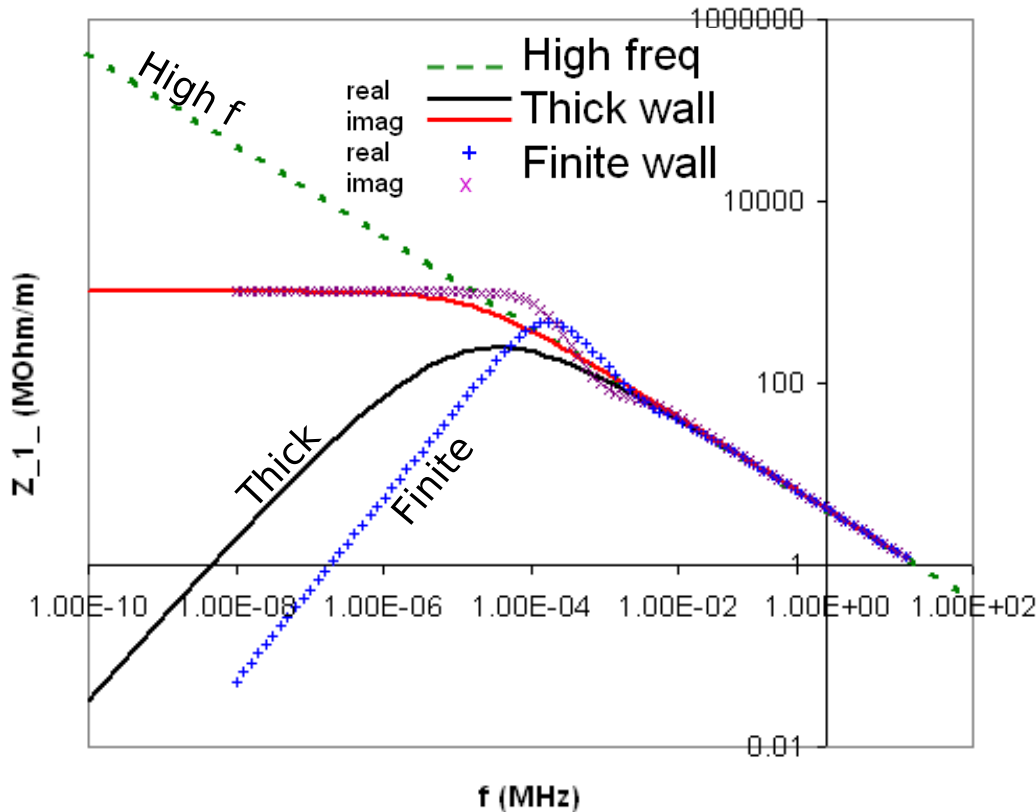
- For small skin depth (δ), the wave in the wall sees a flat wall.
 - This gives an exponentially decaying solution of the electric field in the wall (Chao 1993), leading to an impedance ($1/\sqrt{\omega}$) that diverges at low frequency.
 - For skin depth comparable to pipe radius, the exact solution gives finite impedance at low frequency.



- For infinitely thick wall, there is only outgoing wave in the wall. For finite thickness, this wave is reflected at the outer wall. This can lead to multiple reflections for certain wavelengths.

Impedance for 2 mm wall

2 cm pipe radius, 5 GeV electrons



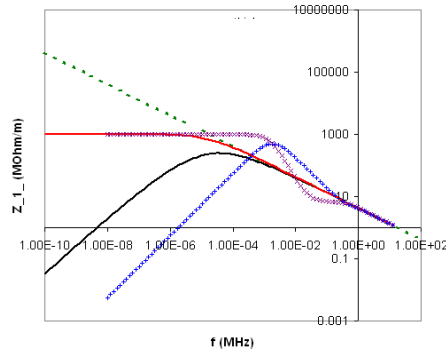
| | <u>Thick</u> <u>wall</u> | <u>Small</u> <u>Skin depth</u> |
|------------------|-----------------------------|-----------------------------------|
| High freq limit | ✓ | ✓ |
| Thick wall limit | ✓ | ✗ |
| Finite wall | ✗ | ✗ |

- Impedance has the correct high frequency limit (which takes the form $1/\sqrt{\omega}$).
- The exact solution for thick wall limit is finite at low frequency.
- Finite wall impedance peaks when half wavelength in wall equals the wall thickness.

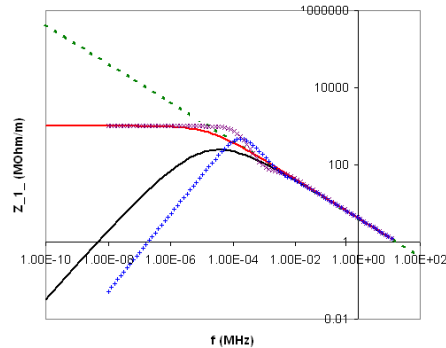
The effect of wall thickness

Wall thickness

0.2 mm



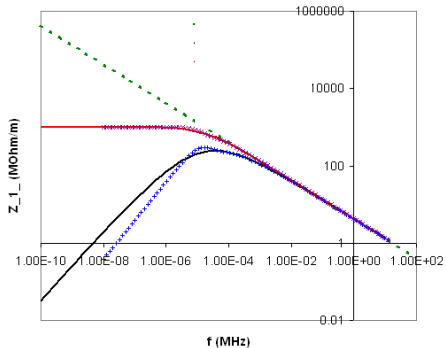
2 mm



real
imag
real
imag

--- High freq
— Thick wall
+ Finite wall
x

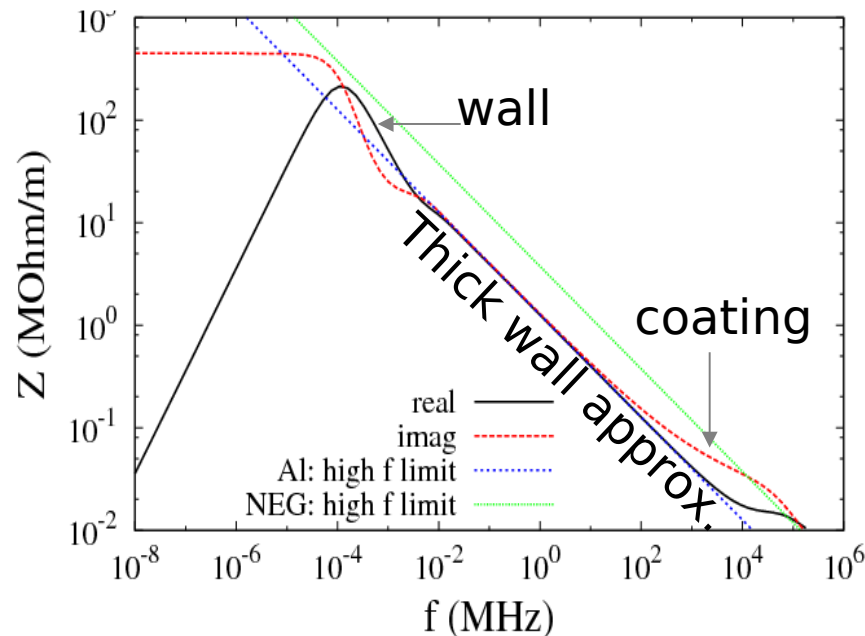
20 mm



- Peak occurs when half wavelength in wall equals wall thickness.
- This explains the shift to lower frequency as wall thickness increases.
- Impedance approaches the correct thick wall limit.

Impedance with NEG coating

- The impedance for a 2 mm thick Aluminium wall and 1 μm thick NEG coating is calculated.

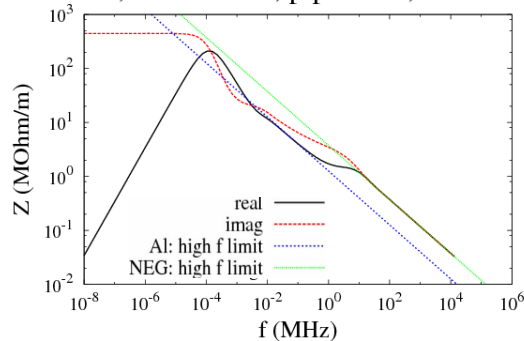


- Frequency domain shows distinct effects of wall and coating
- At low frequency, very different from thick wall approximation
- At high frequency, behaviour switches from Aluminium to NEG because skin depth falls below NEG thickness

The effect of NEG coating

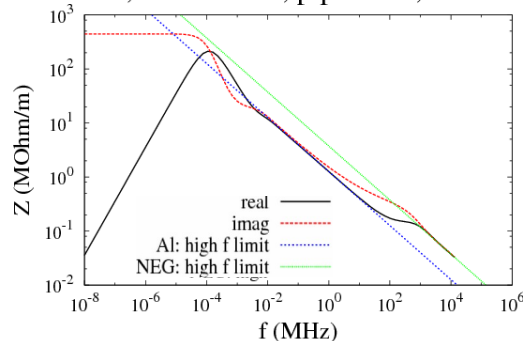
An additional peak or “bubble” appears when half wavelength in coating equals thickness of the coating.

wall 2 mm, beam 3 mm, pipe 3 cm, NEG = 100 μm



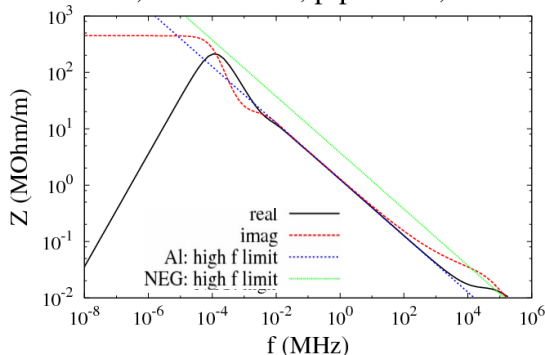
This explains the shift of the “bubble” to higher frequency as coating thickness decreases.

wall 2 mm, beam 3 mm, pipe 3 cm, NEG = 10 μm



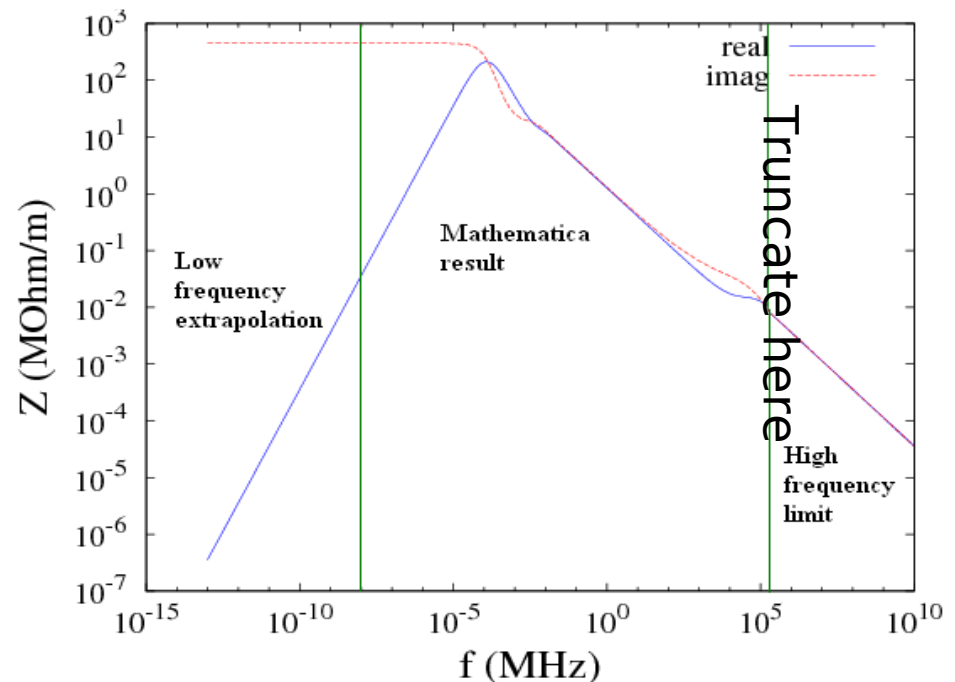
When skin depth in coating becomes much smaller than the coating thickness, impedance switches to the high frequency limit of NEG.

wall 2 mm, beam 3 mm, pipe 3 cm, NEG = 1 μm



Limitations of impedance calculation

- For the beam pipe used, computation fails above 10^5 MHz due to badly conditioned matrix.
- This misses part of the NEG coating “bubble”, and is approximated by the high frequency limit.
- At low frequency, impedance appears to be linear with frequency (needs to be verified analytically).
- These will affect the wake function calculation to follow.



Analytic solution as a benchmark

- Analytic solution is possible at the high frequency limit. The resulting impedance has the form $1/\sqrt{\omega}$.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X_1(\omega) e^{i\omega t} d\omega = x(t)$$

$$X_1(\omega) = \begin{cases} \frac{1-i}{\sqrt{\omega}}, & \omega > 0 \\ -\frac{1+i}{\sqrt{-\omega}}, & \omega < 0 \end{cases} \rightarrow x(t) = \begin{cases} 0, & t > 0 \\ -\frac{2i}{\sqrt{-t}}, & t < 0 \end{cases} \quad t \rightarrow z/c$$

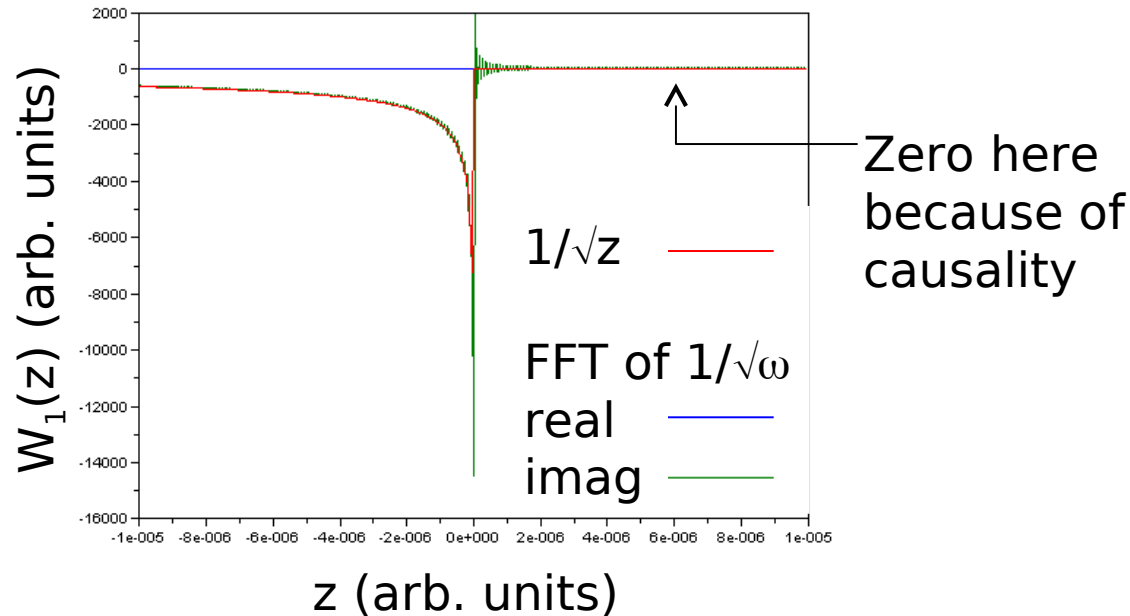
Impedance

Wake
function

- The wake function is obtained by Fourier transform. This can also be done analytically, and gives the same function form of $1/\sqrt{z}$.

Controlling the numerical errors

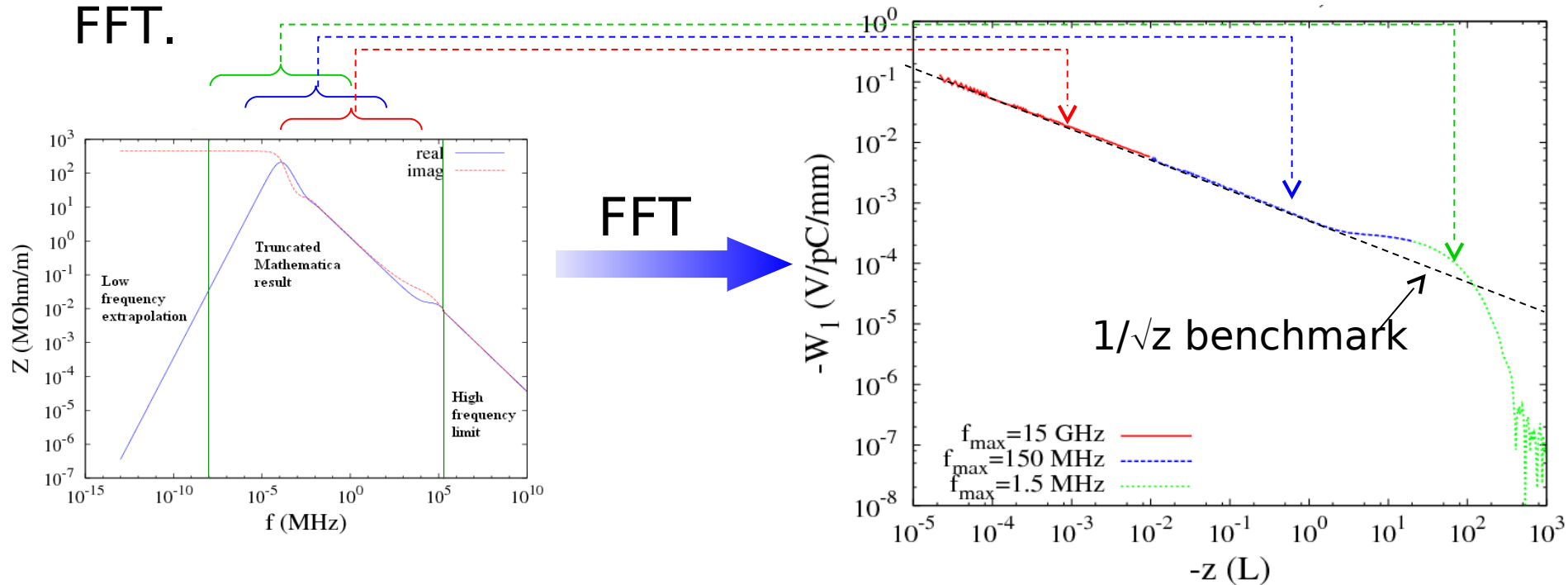
- Deriving the wake function analytically provides a benchmark for the finite wall calculation.



- The Fourier transform must be truncated at some finite integration limits when computed numerically using FFT.
- This introduces errors. Comparison with the $1/\sqrt{z}$ formula tells us the size of this error.

Computing the wake function

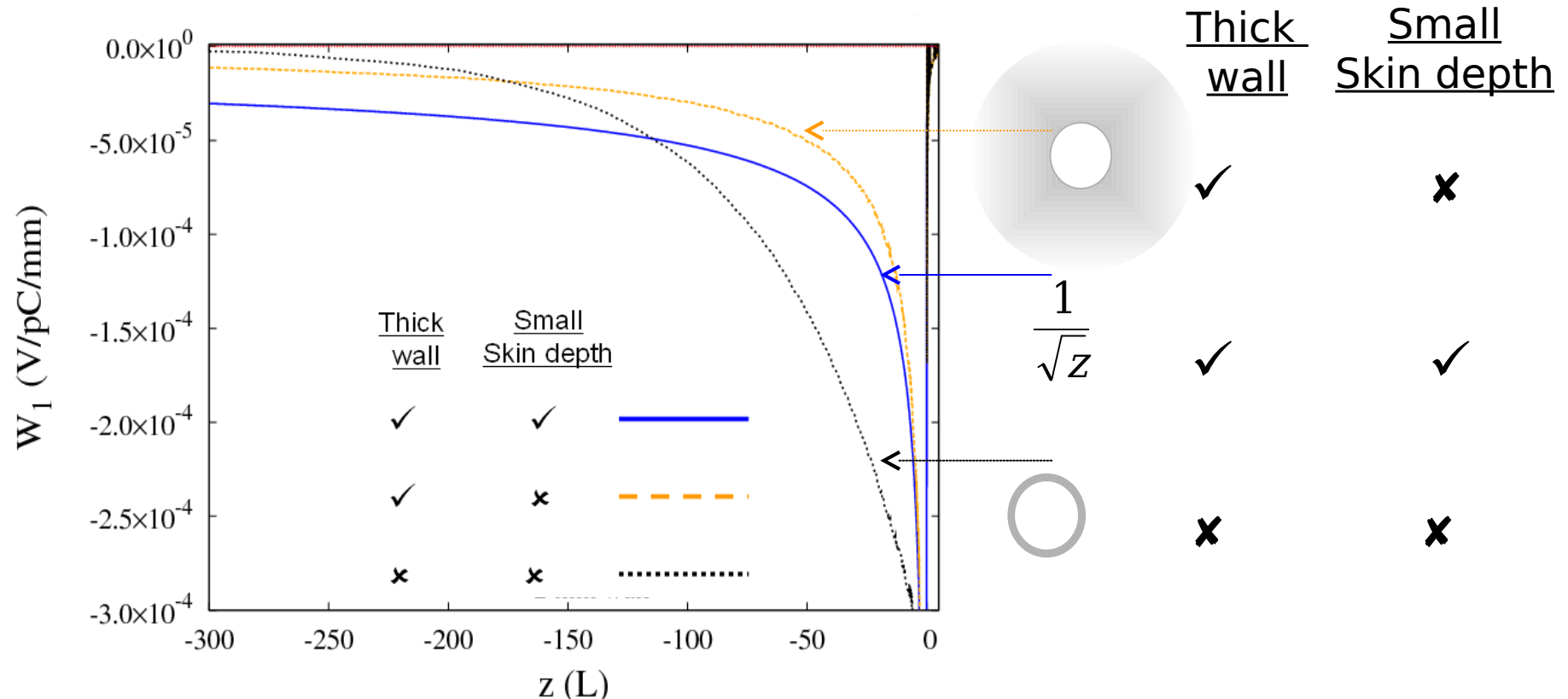
- The wake function is obtained from the impedance using FFT.



- The range of z includes very small and very large values. A straightforward FFT requires a very large array of impedance values – about 50 GB for the ILC damping ring.
- To get around this problem, the FFT may be carried out on smaller arrays for different ranges of z .

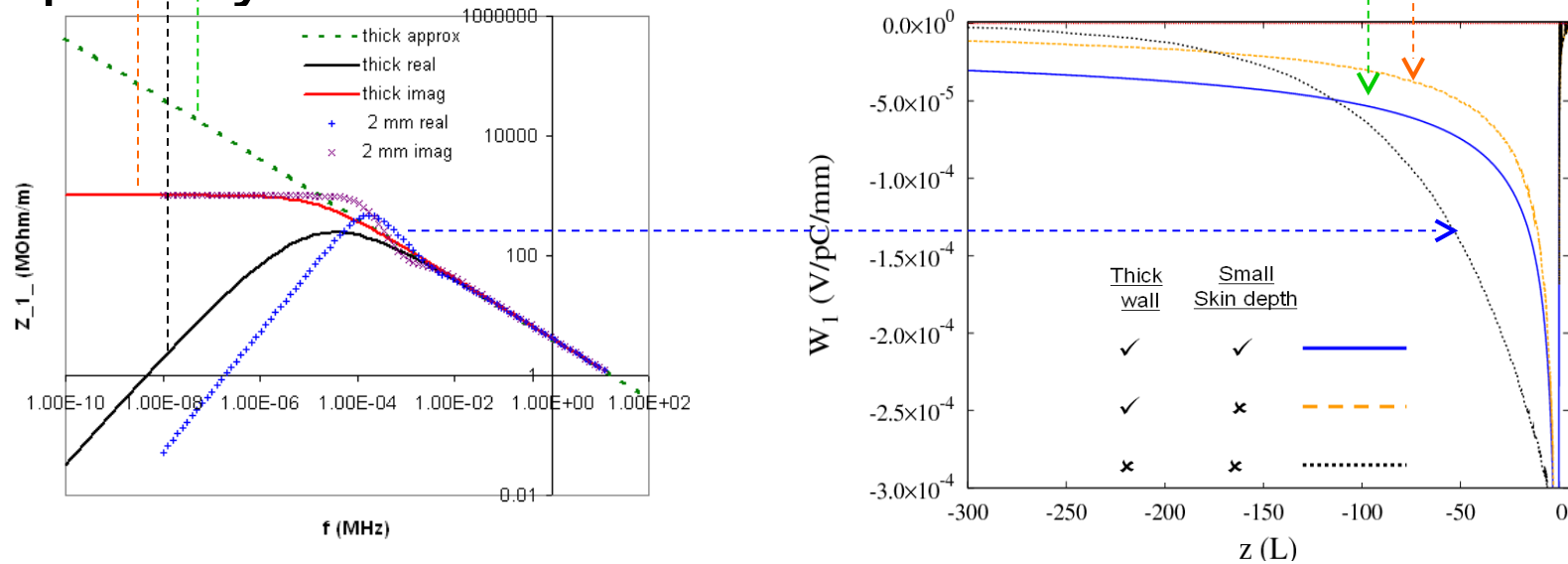
The effect of finite wall

- The results here show the changes as the assumptions are lifted one by one.



Explaining the wake function behaviour

- At large distance, the thick wall wake function is smaller in magnitude than the high frequency limit, because its impedance is finite at low frequency.

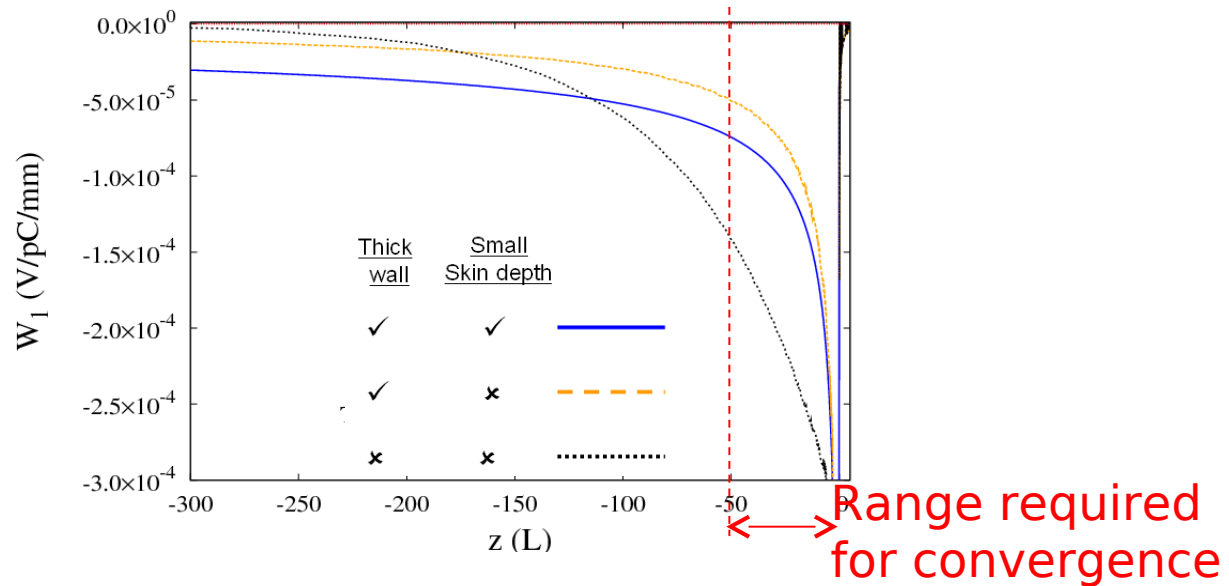


- The finite wall wake function is larger at small distance because of the peak in its impedance.

Significance of finite wall result

$$M d^2 y_m / dt^2 = -M \omega_\beta^2 y_m - q^2 / L [W_1(-c\tau) y_{m+1} + W_1(-2c\tau) y_{m+2} + \dots]$$

- The number of terms required for convergence of the wake force corresponds to 50 turns in the damping ring.



Here, the finite wall result is significantly larger than the high frequency limit. This would have an impact on

Future Plan

- To study transient effects on beam jitter, during injection and extraction.
- Validate the finite wall wake function and calculated growth rates with experiments.
- Include bends in beam pipe, coupling between vertical and horizontal motions, unequal bunch spacings, other ring components, nonlinear effects, ...
- To meet the challenges of new physics regimes in the next generation of accelerators.